Unsteady Flow with Shock Waves: Application to Gasdynamic Lasers

Harold W. Gale*
Air Force Systems Command, Andrews Air Force Base, Md.

Unsteady flow of a compressible gas in a channel of varied cross section is analyzed by the method of characteristics with provisions for an indefinite number of shock waves and a contact surface. A procedure to adjust for boundary-layer displacement thickness growth and a method of application to gasdynamic lasers (GDL) are described.

Nomenclature

A = cross-sectional area
c = speed of sound
F = friction coefficient
f = function
h = enthalpy
M = Mach number
p = pressure
q = energy transfer coefficient
R = gas constant
s = entropy
T = temperature
t = time
u = velocity in x direction
$\bar{v}$ = velocity vector
v = velocity magnitude
x = axial coordinate
γ = specific heat ratio
ρ = density
∇ = vector del operator

Subscripts

i = specific initial-value line or solution line
0 = stagnation value
t,x = partial derivatives with respect to the variable
$1,2...$ = item or point designation

I. Introduction

THE development of flow from initial conditions is of great interest in the start transient of rockets, gasdynamic lasers, and similar apparatus. Small changes in flow channel geometry because of boundary-layer growth or other causes can result in significant changes in the unsteady flow and the end-state flowfield.[1,2] Krein[3] recently obtained similar results. The analytical procedures described in this paper were developed in conjunction with the efforts by Krein.

A method-of-characteristics technique for one-dimensional flow with an indefinite number of discontinuities in a varied-area channel was developed. The technique as a concept was proposed by Hartree[4] before modern computers were available. In the method described here, the solution points are determined by the intersection of right-running wave characteristics and specified constant time lines. The algorithm permits the formation of an indefinite number of right-running shocks and left-running shocks in the flowfield.

The analysis of complex unsteady flows in channels of varied cross-section has become feasible with modern computer capability. Of the various numerical analysis methods, the method of characteristics is most suitable for unsteady flows with multiple gasdynamic discontinuities. According to Roache,[5] the direct solution of the governing equations by finite-difference methods has not reached a comparable stature in the treatment of an indefinite number of discontinuities. The direct solution methods smear discontinuities to form a zone of many solution points. The width of the zones tends to increase with time, or diffuse, because of mathematical features of the analytical procedure. The amount of smearing usually is controlled by the mathematical introduction of "artificial" viscosity. Several discontinuities can be treated rigorously by patching techniques if the discontinuity origins are known from the problem definition.[6] In contrast, the method of characteristics readily includes and retains an indefinite number of discontinuities.

The method of characteristics has had considerable use in the analysis of multidimensional steady flows and in the analysis of relatively smooth unsteady one-dimensional flows. The method of characteristics can be used, since the governing partial differential equations are hyperbolic in such flow systems. The principal difficulties arise in the treatment of boundaries and discontinuities in the flowfield. Very little research apparently has been reported on the rigorous treatment of boundary conditions. Since the boundary conditions, except the channel flow area, are coupled to the development of the central flow, a necessary first step is the successful modeling of the central flow. Later extensions of the analysis could include models for energy transfer, wall friction, venting, boundary-layer processes, and reacting flow.

II. Assumptions and Basic Equations

The equations of motion for the unsteady one-dimensional flow of an inviscid fluid are well established. Those equations can be classified as a system of quasilinear first-order hyperbolic partial differential equations. The number of dependent variables depends on the assumed nature of the flow, that is, whether or not the chemical composition is frozen or chemical equilibrium exists, number of chemical components, etc. The mathematical character and the theoretical method of solution are independent of such specific assumptions. The specific numerical algorithm and the associated computer program, however, depend on the particular system of assumptions and equations.

The physical nature of elementary unit processes, such as elementary waves and shocks, is well established for steady flows. The behavior of such unit processes during strongly unsteady flow are not well documented: for example, only two studies of shock strengthening in conical convergent channels were identified.[7,8] Because of inadequate physical understanding of the interactions of the unsteady turbulent

Presented as Paper 76-743 at the AIAA/SAE 12th Propulsion Conference, Palo Alto, Calif., July 26-29, 1976; submitted Aug. 16, 1976; revision received Feb. 22, 1977.

Index categories: Nozzle and Channel Flow; Nonsteady Aerodynamics.

*Lieutenant Colonel, U.S. Air Force. Member AIAA.

boundary with shock waves, the boundary-layer process was not included in this analysis.

The continuity of one set of wave characteristics was achieved by placing the solution plane spatial points on the selected wave characteristics. The right-running wave characteristics were chosen, and the gas source was placed at the left end of the channel. Thus, flow would develop from left to right, and the right-running waves would be the dominant characteristics.

The particle path characteristics and the left-running wave characteristics are evaluated by interpolation, as in the original Hartree method. The major advantage of the specified time interval continuous characteristic inverse method of characteristics (Hartree-wave method) is the reduction of smearing and interpolation uncertainty in the evaluation of the dominant characteristics. For a wave system entering an undisturbed field, the leading-edge wave experiences no smearing or diffusion. Within the disturbed field, the left-running elementary waves are diffused. By following the particle path characteristic, the scheme was used to analyze the contact surface motion. The simple procedure outlined by Hartree for analysis of shocks was used unchanged.

Governing Equations

In general, the flows of interest can be treated as consisting of isentropic interior regions patched together at discontinuities such as shocks. Noncharacteristic relations such as the Rankine-Hugoniot relations are used at the discontinuities.

The motion of many compressible fluids can be described accurately by the governing equations for an adiabatic inviscid flow. The main assumptions for the interior region gasdynamic model are as follows: 1) continuum, 2) adiabatic, 3) isentropic, 4) inviscid, 5) simple thermodynamic system, and 6) smooth initial data and boundaries.

The governing equations in differential form are as follows:

Continuity

$$\rho_t + \nabla \cdot (\rho \bar{v}) = 0 \tag{1}$$

momentum

$$\rho (\mathrm{D}\bar{v}/\mathrm{D}t) + \nabla p = 0 \tag{2}$$

Energy

$$\mathrm{D}(h + v^2/2)/\mathrm{D}t - p_t = 0 \tag{3}$$

Entropy

$$\mathrm{D}s/\mathrm{D}t = 0 \tag{4}$$

Since flow along particle paths (path lines) is isentropic between discontinuities, the energy and entrophy equations can be replaced by an equivalent equation, where $c = c(p,\rho)$ for a simple thermodynamic system:

$$\mathrm{D}p/\mathrm{D}t - c^2 \mathrm{D}\rho/\mathrm{D}t = 0 \tag{5}$$

Expanding these equations for unsteady one-dimensional flow in the x direction gives the following set of equations:

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{6}$$

$$\rho u_t + \rho u u_x + p_x = 0 \tag{7}$$

$$p_t + up_x - c^2 \rho t - c^2 u \rho_x = 0 \tag{8}$$

Equations (6-8) are the equations governing the unsteady one-dimensional flow in the interior regions between the discontinuities and in the regions between the boundaries and the discontinuities. When shocks occur, the property distribution ceases to be continuous and isentropic. The governing equations across shock waves are developed from the Rankine-Hugoniot relations. The flow, however, was not restricted to constant particle stagnation enthalpy. Equation (3) shows that when p_t is not zero the stagnation enthalpy and stagnation temperature will change.

Gas products from combustion of well-mixed fuel and oxidizer form a simple system in the stagnation state. Specification of two thermodynamic properties is sufficient to define that state. Gas generator stagnation pressure and stagnation temperature are the most convenient choices for a GDL. For a thermally and calorical perfect gas, the density and speed-of-sound relations are given by

$$p = \rho R T \tag{9}$$

$$c = \sqrt{p/\rho} \tag{10}$$

The set of differential equations (6-8), together with the equation for the speed of sound, form a complete set of first-order, quasilinear, partial differential equations.

The characteristic relations can be generalized to cylindrical and spherical coordinated by the addition of one term to the compatibility relations of the wave characteristic, to yield the following results[9]:

Right-Running Waves

$$\mathrm{d}p + \rho c \mathrm{d}u + (\rho c^2 \nu u/x)\mathrm{d}t = 0 \tag{11}$$

Left-Running Waves

$$\mathrm{d}p - \rho c \mathrm{d}u + (\rho c^2 \nu u/x)\mathrm{d}t = 0 \tag{12}$$

Particle Paths

$$\mathrm{d}p - c^2 \mathrm{d}\rho = 0 \tag{13}$$

For Cartesian coordinates $\nu = 0$, for cylindrical coordinates $\nu = 1$, and for spherical coordinates $\nu = 2$. A singularity exists at the spatial origin.

The influence of area change, friction, and energy transfer acting on the flow can be considered by means of nonhomogeneous terms added to the wave characteristic compatibility equations as follows.[10,11] Along right-running waves,

$$\mathrm{d}u + \frac{\mathrm{d}p}{\rho c} + \frac{c}{A}\left[\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\mathrm{d}A}{\mathrm{d}t}\right]\mathrm{d}t - \frac{(\gamma-1)q}{c}\mathrm{d}t + F\left[1 - (\gamma-1)\frac{u}{c}\right]\mathrm{d}t = 0 \tag{14}$$

Along left-running waves,

$$\mathrm{d}u - \frac{\mathrm{d}p}{\rho c} - \frac{c}{A}\left[u\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\mathrm{d}A}{\mathrm{d}t}\right]\mathrm{d}t + \frac{(\gamma-1)q}{c}\mathrm{d}t + F\left[1 + (\gamma-1)\frac{u}{c}\right]\mathrm{d}t = 0 \tag{15}$$

The area change term is applicable to Cartesian coordinates only.

The characteristic equations were integrated by means of the modified Euler method. Correct treatment of the boundary conditions was crucial in the accuracy of the solutions.[12] Boundary conditions at the channel left end simulated shock source inflow, closed end, and ramp pressure increase with time. Initial channel conditions could be specified. Boundary conditions at the channel right end accommodated choked or subsonic inflow, outflow at all Mach conditions, and pressure decay behind an exiting shock wave.

III. Accuracy Studies

General

The accuracy of an unsteady flow numerical algorithm is defined by the order, by the precision of wave propagation, by the property distributions, and by comparison to experimental results.

Order

A field with a hyperbolic property distribution can lead to the existence of truncation errors that are first order but nonlinear with distance. Since the general case of concern in this research is a channel of varied cross-section and such variations of cross-section lead to highly nonlinear property distributions, the overall algorithm is considered to be first order.

Wave Propagation

Small changes in pressure can be propagated by isentropic acoustic waves. In a tube of constant diameter, a small step increase in pressure should propagate without change in shape or speed. In a channel of conical geometry, the magnitude of the pressure step would decrease as a hyperbolic[11] function of distance. A shock wave should propagate at constant velocity in a constant-diameter tube. These results are obtained with this algorithm.

Steady Flow

The convergence of the algorithm to an accurate steady-state property distribution was examined. The initial conditions were 100 psi stagnation pressure at the nozzle entrance, 1.0 psia at the nozzle exit, and a uniform stagnation pressure of 15 psia, with a uniform Mach number of 1.22 within the nozzle. Convergence to steady-state values was obtained in approximately 180 time steps for a 60-point array. The numerical analysis results are presented with the theoretical curve in Fig. 1.

Comparison with Experiments on Shock Strengthening and Reflection

One of the critical features in the analysis of transients is the correct prediction of shock interaction with a highly contracted channel such as a diffuser. Setchell et al.[8] published the results of a pertinent series of experiments in which Mach 6 shocks were reflected from a 10° half-angle conical convergent channel section mounted on the end of a pressure-driven shock tube. The contraction ratio was approximately 2300 to 1. The gas used in the experiments was argon at a pressure of 1.5 Torr. Setchell compared the results to the predictions of several theoretical formulations.

In the present numerical analysis study, the results were obtained for argon at a pressure of 1 psia in a tube geometry as described by Setchell, except that no allowance was made for the instrumentation blockage of the channel. The apex of the simulated channel was open, with an orifice radius of 0.012 in. The shock-tube radius was 3.010 in. The contraction ratio obtained in the numerical analysis model was about 6300. The additional blockage due to instrumentation would cause the analytical results to be shifted slightly toward the apex as compared to the experimental results. The flow source was assumed to be invariant (the shock source boundary condition).

The numerical analysis predictions of the shock velocity during entry to the cone are plotted in Fig. 2 with Setchell's results. The experimental data "steps" were three-dimensional reflected wave crossings. The analytical prediction used 290 points in the initial array and about 35 time steps. At the last time step, the array contained 29 points. The agreement of experimental and analytical results was considered satisfactory in view of the extremely complex multidimensional nature of the physical process.

The experimental reflected shock behavior was quite at variance from the predictions, since the velocity was irregular near the apex of the cone and nearly constant near the entrance. Setchell compared the results of the Guderley similarity formulation with the experimental data. The numerical results are plotted with Setchell's data in Fig. 3. The numerical array contained 33 points initially, and 38 time steps were used. The array contained 48 points at the last time step. It is noted that the numerical analysis results deviate from the Guderly formulation results in a manner closely resembling the trend of the experimental data. The numerical analysis results are in satisfactory agreement with the experimental data.

IV. Application to Gasdynamic Lasers (GDL's)

General

Start of a GDL is defined as the achievement of steady supersonic flow in the laser cavities. If the diffusers used a single normal shock wave at the diffuser inlet contraction ratio of less than 1.5, the stagnation pressure loss would be 86% with Mach 4 velocity in the laser cavity. In contrast, aircraft inlet diffusers using multiple oblique shocks can obtain more than 50% stagnation pressure recovery with a contraction ratio up to about 5.3 in operation.

A contraction ratio greater than that permitted for a single normal shock usually would prevent start in a fixed-wall nonvented diffuser, with a gradual pressure increase.

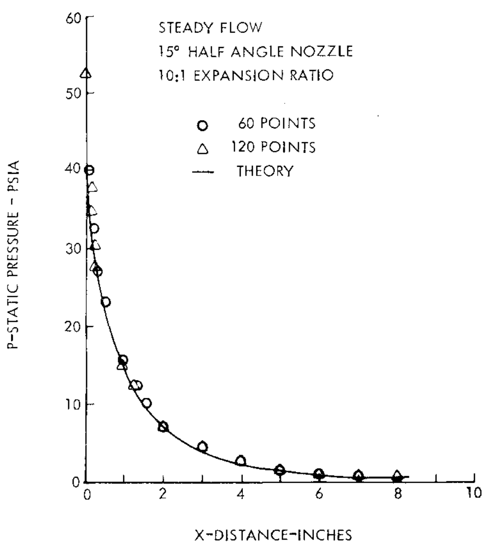

Fig. 1 Steady flow pressure profile.

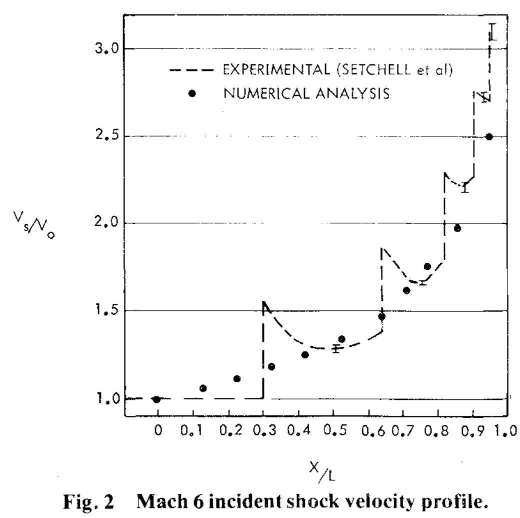

Fig. 2 Mach 6 incident shock velocity profile.

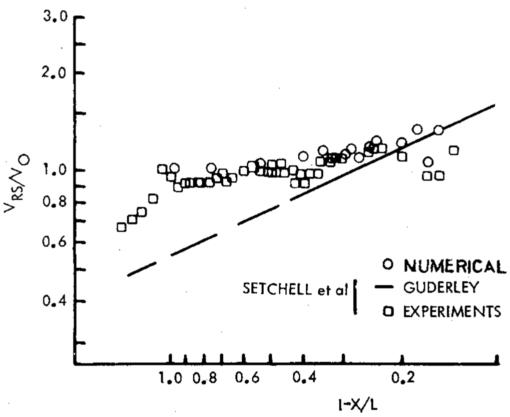

Fig. 3 Reflected shock-wave velocity.

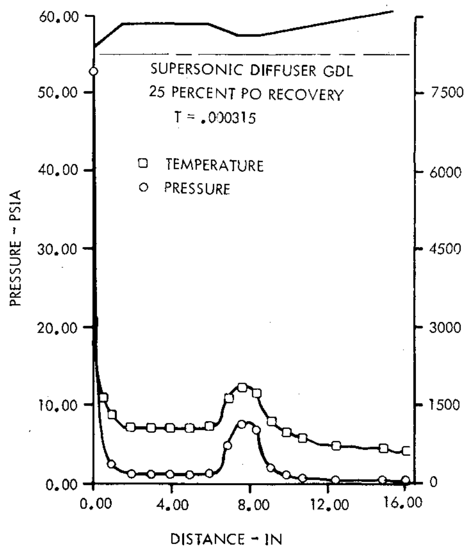

Fig. 4 Supersonic initial velocity.

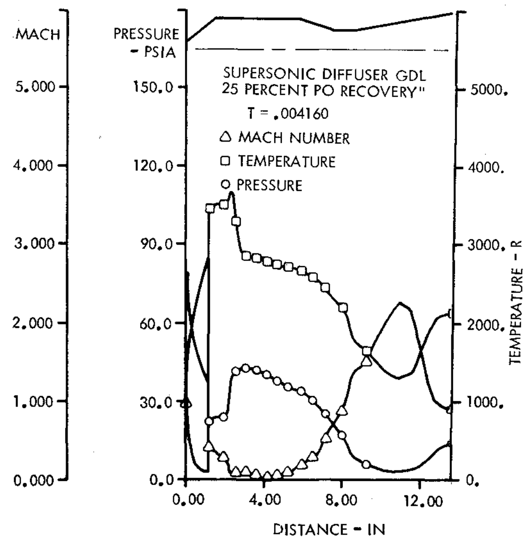

Fig. 5 Subsonic initial velocity.

However, fixed-geometry diffusers with a contraction ratio of 4.5 have been started by means of unsteady wave techniques, with entry velocities of Mach 4 and about 10% mass bleed. If unsteady wave processes could be used to start GDL's incorporating fixed-geometry diffusers, it appeared that staged GDL's would be feasible. At least two versions of staged GDL's were proposed in the literature.[13]

The problem of determining criteria for starting staged GDL's was approached in several steps. The first step was identification of a critical condition of the initial flowfield which would permit start. The second step was evaluation of plausible techniques for obtaining the critical condition.

Critical Initial Conditions for Stable Operating Modes

Plausible channel geometries were examined with initial conditions of uniform supersonic flow throughout. The trend of subsequent flow development then was observed to determine whether an unstarted or a started condition was developing.

Every configuration examined would start if the initial velocity was supersonic throughout. Figure 4 illustrates the very orderly property field typical of the starting process, at 0.000315 sec after the onset of flow development. Only configurations with a contraction ratio much less than the normal shock-wave criterion would start smoothly with subsonic initial velocity throughout. Figure 5 illustrates the complex flowfield that developed during the failure to start of a 2.50 contraction ratio configuration. A region of reversed flow was regarded as a sufficient indication of failure to start. Similar results were obtained from the subsonic diffuser staged GDL's.

In subsonic flow, the diffuser inlets reflect the pressure waves upstream. The strength of the reflected waves depends on the contraction ratio, as it is noted that a closed end corresponds to a contraction ratio of infinity. The reflected waves can reverse the flow in the laser cavity temporarily if the pressure rise rate and the contraction ratio are great enough. Moderate pressure rise rates and low contraction ratios would reduce the strength of the reflected waves. Reflected waves tend to combine with and augment the normal shock wave standing in the GDL nozzle, to slow the process of starting. The criterion for start of single normal shock diffusers would apply, however, and a configuration with a sufficiently small diffuser contraction ratio would start eventually in the absence of viscous effects.

From the study of operating modes of staged GDL's, it was concluded that starting of such devices by one-dimensional unsteady processes would require establishing supersonic flow in a single step. A critical condition exists for an initial velocity near Mach 1.0.

Start Procedure

The study of GDL stable operating modes indicated that high-performance fixed-geometry unvented diffusers could be started by unsteady wave processes if the flow was initially everywhere supersonic behind the start pulse leading edge. A very rapid pressure rise in the rocketlike gas generator of a GDL would exert a "piston" driving force on the gases initially present in the channel. The limiting velocity of the driving gas in a constant-diameter channel is the speed of sound in the driving gas, since choking occurs at the channel entrance. Several experiments were performed with constant-diameter channels to evaluate the distance required to form a shock wave and to evaluate the resulting property distributions.

A sample graph is presented (Fig. 6). The gas generator stagnation pressure rise was 50,000 psi/sec. The gas generator stagnation temperature was constant at 3600°R. These values were thought to be representative of the upper limits that could be anticipated in GDL's. The shock wave was observed to form within about 4 in. of the channel entrance, but the maximum velocity, near Mach 1, was achieved only after the shock had traveled about 30 in. Onset of choking of the driver gas at the entrance tended to reduce the rate of development of the leading-edge shock. In every case examined, the velocity behind the contact surface, in the driver gas, was subsonic.

It was concluded that the start conditions and geometries of GDL's probably are not compatible with generation of a start pulse that has initially everywhere supersonic velocity behind the leading-edge pulse. The high-expansion-ratio nozzles typical of GDL's accelerate and weaken pressure waves, as demonstrated in the accuracy study. This process is the reverse of the action required. The diffusers retard and strengthen the pressure waves in subsonic flow but with the consequence of reflecting pressure waves upstream. The

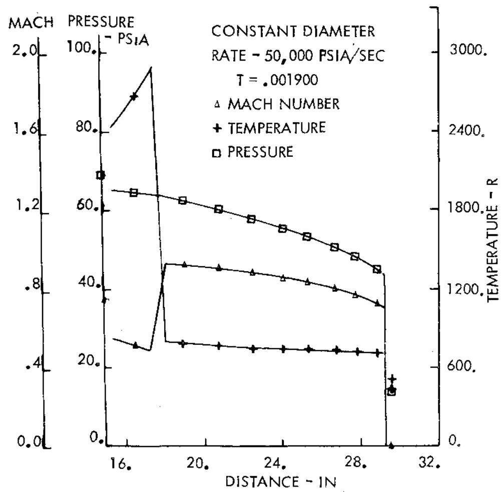

Fig. 6 Developed property profiles.

reflected waves further restrict the development of supersonic flow in the nozzles.

Viscous Interactions

Repeated comments have been made concerning the significance of the boundary-layers in controlling the flow, as confirmed in the experimental apparatus. Krein's initial experimental model simulated a single-stage GDL. The channel had a circular cross-section, a single contoured nozzle, and a single conical entrance diffuser. The diffuser had a 5° conical convergent inlet about 2 in. long, a constant-diameter section with a length-to-diameter ratio of 10, and a short flared extension.

The first experiments were performed with a diffuser contraction ratio of 1.34, which is more open than the ideal contraction ratio of 1.41 for an ideal normal shock at the design Mach number of 3.2. The original configuration would not start with any available reservoir stagnation pressure. Elimination of the constant-diameter portion of the diffuser or use of boundary-layer venting at the diffuser inlet, or use of a burst diaphragm ahead of the nozzle permitted diffuser start at reservoir stagnation pressures above 150 psig. The pressure profiles in the started mode were consistent with flow separation in the nozzle near the nozzle exit. In the started mode, supersonic flow at the diffuser entrance was indicated by increase of static pressure along the diffuser channel.

When the diameter of the full-length diffuser was increased by 0.076 in., from 3.200 to 3.276 in. (1.28 contraction ratio), the diffuser started at 53 psig reservoir pressure. With decrease of reservoir pressure, the diffuser unstarted at 37 psig. In the original configuration, an arbitrary but insufficient allowance had been made for viscous losses. In the modified design, an allowance was made for the boundary-layer displacement thickness, added to the ideal radius of the diffuser channel, as recommended by McLafferty. Subsequently, a rectangular-cross-section model was constructed with a 100% factor of safety on boundary-layer blockage of the channel. The rectangular-section channel started at about 210 psig by means of an ejectorlike process.

Boundary-layer analysis was performed by means of the ICRPG Turbulent Boundary Layer Nozzle Analysis Computer Program.[14] The program indicated that flow separation was likely in the subsonic zone of the circular channel nozzle inlet. Flow separation was indicated downstream of any strong normal shocks. Rapid thinning or "relaminarization" of the boundary layer was indicated for regions of sustained strong acceleration of the flow. The experimental results were generally in satisfactory agreement with the computer indications. The computer program did not evaluate the stagnation pressure losses (viscous losses) due to the boundary layer. McLafferty had reported that the losses were proportional to the fractional channel blockage due to the boundary-layer displacement thickness, up to about 10%.

The study of viscous interactions demonstrated the complexity of such processes. Available numerical analysis procedures and understanding of the physical processes were not adequate for a useful algorithm combining analysis of the central flow and analysis of the boundary layer.

V. Comments and Summary

Numerical Analysis

A procedure for analysis of unsteady one-dimensional flows with an indefinite number of discontinuities has been demonstrated. The overall procedure is complex, but no new knowledge of the inviscid physical processes was required. The accuracy of the analysis can be of the order of 1.0%. The computer program code and sample cases are the subject of a separate report.[15]

GDL Start Transient Analysis

Unsteady wave start of fixed-geometry high-performance diffuser staged GDL's is probably not feasible if one-dimensional inviscid processes are dominant. A sufficiently strong continuous start pulse probably cannot be obtained with plausible fixed-geometry GDL channels. Diffusers designed for the stagnation pressure ratios typical of normal shock waves possibly would start with adequate boundary-layer control.

Acknowledgments

This research was performed under contract to the U.S. Army Missile Command, Redstone Arsenal, Ala., Contract No. DAAH 01-72-C-0864. The advice and assistance of Professor J.D. Hoffman, Mechanical Engineering Department, Purdue University, are gratefully recognized.

References

[1]Shapiro, A.H., *The Dynamics and Thermodynamics of Compressible Fluid Flow,* Vols. I and II, Ronald Press, New Yor, 1953.

[2]McLafferty, G., "Theoretical Recovery Through a Normal Shock in a Duct with an Initial Boundary Layer," *Journal of the Aeronautical Sciences,* Vol. 20, March 1953, pp. 169-174.

[3]Krein, S.W., "An Experimental Investigation of the Starting Characteristics and Diffuser Performance of Gasdynamic Laser Systems," M.S. Thesis, Purdue Univ., 1973.

[4]Hartree, D.R., "Some Practical Methods of Using Characteristics in the Calculation of Nonsteady Compressible Flow," U.S. Atomic Energy Commission Rept. AECU-2713, 1953.

[5]Roache, P.J., *Computational Fluid Dynamics,* Hermosa Publishers, Alburquerque, N. Mex., 1972.

[6]Moretti, G., "A Critical Analysis of Numerical Techniques: The Piston-Driven Inviscid Flow," Polytechnic Inst. of Brooklyn, PIBAL Rept. 69-25, July 1967.

[7]Russell, D.A., "ShockWave Strengthening by Area Convergence," *Journal of Fluid Mechanics,* Vol. 27, 1967, p. 305.

[8]Setchell, R.E., Storm, E., and Sturtwant, B., "An Investigation of Shock Strengthening in a Conical Convergent Channel," *Journal of Fluid Mechanics,* Vol. 56, Pt. 3, 1972, pp. 505-522.

[9]Chou, P., Karpp, R.R., and Huang, S.L., "Numerical Calculations of Blast Waves by the Method of Characteristics," *AIAA Journal,* Vol. 5, April 1967, pp. 618-623.

[10]Rudinger, G., *Wave Diagrams for Nonsteady Flow in Ducts,* Van Nostrand, New York, 1955.

[11]Gale, H.W., "Numerical Analysis of Unsteady Compressible Gas Dynamics of Varied Area Channels: Applied to Gas Dynamic Laser Start," Ph.D. Thesis, Purdue Univ., 1974.

[12]Blitz, J., *Elements of Acoustics,* Butterworth, Washington, D.C., 1964, pp. 34, 55, 56.

[13]Jones, T. and Neice, S., "Starting and Pressure Recovery in Fixed Throat Supersonic–Supersonic Diffusers," Mathematical Sciences, Northwest, Inc., MSNW Rept. 71-66-4 (AFWL-TR-71-83), Seattle, Wash., 1971.

[14]"ICRPG Turbulent Boundary Layer Nozzle Analysis Computer Program," Interagency Chemical Rocket Propulsion Group, Army-Navy-Air Force-NASA Performance Standardization Working Group, Pratt & Whitney Aircraft Div., United Aircraft Corp., East Hartford, Conn., July 1968.

[15]Gale, H.W., "Numerical Analysis of Unsteady Compressible Gas Dynamics of Varied Area Channels: Applied to Gas Dynamic Laser Start," *Computer Program User's Manual,* 1974.